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Research Article

Reducing Variation of Risk Estimation by Using Importance Sampling

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ABSTRACT

In today's world, risk measurement and risk management are of great importance for various economic reasons. Especially in the crisis periods, the tail risk becomes very important in risk estimation. Many methods have been developed for accurate measurement of risk. The easiest of these methods is the Value at Risk (VaR) method. However, standard VaR methods are not very effective in tail risks. This study aims to demonstrate the usage of delta normal method, historical simulation method, Monte Carlo simulation, and importance sampling to calculate the value at risk and to show which method is more effective by applying them to the S&P index between 1993 and 2003.

Keywords:

Importance Sampling, Value at Risk, Monte Carlo Simulation, Delta Normal Method, Tail Risk



1. Introduction

Since the beginning of human life, the instinct of protection against external factors has evolved. This development has been integrated into every aspect of human life and has become increasingly widespread. One of the mentioned areas and perhaps the first thing that comes to mind is the area of finance. As in all areas, the risk is very important in the financial markets. The reason for this importance is the need to estimate, if possible, to reduce the risk, regardless of whether it has an institutional or individual identity, and to be prepared for the situation and conditions that risk it as the most natural state.

Between 2007 and 2009, the world saw the biggest financial crisis since the 1930s. In addition to other factors such as the downturn, the liquidity crisis, the credit crisis or the banking crisis, the integrity of many banks and other financial institutions was seriously endangered in this period, and a few of them required external and state support (de Vooy, 2012, pp.2).

The crisis showed the relatively fragile world banking system and these developments required more effective risk management. It was the alarm bell that augmented global awareness toward tail risk among financial risk managers (Gupta and Chaudhry, 2019). Gupta and Chaudhry (2019) also state that the occurrence of negative extreme events is more frequent than suggested by the normal distribution. As stated by Keçeci and Demirtaş (2018), the traditional risk theory works quite well when the return distributions are close to normal. A weakness of this approach is the assumption of a specific probability distribution. Therefore, traditionally used measures of market risk (i.e. variance or standard deviation) might be insufficient to approximate the likelihood of maximum loss that a firm may witness under highly volatile or normal periods. In this context, new techniques and some models for risk measurement have been developed and the most common and easy to use method is the Value at Risk method. Neftci (2000) found that the implied VaR would be 20% to 30% greater if one used the extreme tails rather than following the standard approach and also show that the VaRs calculated using tails of extreme distributions are significantly more precise than the standard approach.

Value-at-risk is a financial instrument that measures the worst expected loss in a given confidence interval under normal market conditions over a period of time. The fact that the loss is given as a single number is often applied to the investors because it is clear and easy (Jorion, 2003). Many of these techniques and models are popular because they are mathematically resolvable with an easy calculation of various risk criteria. More realistic (and complex) models often have a significant calculation cost that requires Monte Carlo methods to estimate the quantities of interest (Brereton et.al, 2012, pp.1).

Efficient estimation of credit risk measurements is often difficult and costly to calculate, as they require small amounts to be estimated (Brereton et.al, 2012, pp.4). In the case of rare event simulations, common Monte Carlo simulation becomes inefficient while the tail probabilities are estimated. Therefore, when Monte Carlo simulation is used to accurately measure credit risk in corporate credit portfolios, the variance of the estimator should be reduced. A commonly used method for this is the importance sampling. The idea behind the importance sampling is that the

determined values of the random variables in a simulation have more effect than the others in the predicted parameter. If these "significant" values are highlighted by more frequent sampling, then the variance of the estimator can be reduced (de Vooy, 2012, pp.23). The main issue in the implementation of the importance sampling simulation is the selection of biased (or shifted) distribution that promotes important regions of the input variables. Therefore, the selection of optimal biased distribution is very important. In the literature, it is often seen that there is normal distribution or uniform distribution with a sampling distribution for the cases where the events of interest are in the normal distribution. This study aims to investigate which VaR(value-at-risk) method is efficient among delta-normal method, historical simulation method, Monte Carlo method and importance sampling method by applying them to S&P index between 1993-2003 and to guide the investors in selection of an appropriate method.

2. Literature

The Var method shows the maximum risk or loss that can occur at a given probability scale and within a certain time frame, within a confidence interval. It usually has a severe asymmetric character as the risk or loss behavior mentioned. As one of the important reasons for this asymmetric character, the correlation between the factors to be measured can be shown.

Due to this situation, Monte Carlo (MC) simulation has an importance in the analysis of credit risk and determination of the behaviors of other financial instruments. Morokoff (2004, pp.1626) stated the importance of MC simulation at the point of revealing the risk and loss characteristics and obtaining the distribution of this behavior.

The mentioned risk and loss are rare situations because they are generally low in terms of probability. Hence, the expectation is a conditional expectation and the cost of calculation is high in terms of process and time, and at the same time increases the variability of the estimation. At this point, the concept of importance sampling, which is one of the techniques developed to reduce the process load that will significantly reduce the variability, is encountered.

The earliest descriptions of the importance sampling were made by Kahn (1950a, 1950b). The most famous result for the distribution of optimal importance (or suggestion) was given by Kahn and Marshall(1953). This technique was first applied by Glasserman et al. (1999a, 1999b) for risk measurement in finance. Glasserman and Li (2005, pp.1650) proposed a method of importance sampling to reduce the variance of loss probabilities, and Glasserman (2005) developed relationship importance sampling estimators to this method proposed by Glasserman and Li(2005) when the loss distribution was discrete. In another study, Bassamboo et al. (2008, p.600) have shown that importance sampling method is asymptotically efficient as compared to MC simulation. Kalkbrener et al. (2004) use importance sampling to study capital allocation for credit portfolios.

When the distribution of losses is continuous, Liu (2010, pp.2774) proposed a method of limited importance sampling, which produced simulated observations to collect in a specific region with probability value. Müller (2016, pp.11), by using the

importance sampling method, revealed that the variance of the loss probabilities decrease and thus more efficient predictors can be obtained.

3. Method

There are four important parameters used in the calculation of value at risk. These are confidence level, lock-up period, distribution of financial assets and portfolio diversification.

Confidence Level: in the literature, 95% and 99% confidence levels are used in VaR calculations with the assumption that returns are normally distributed. Naturally, the increase in the level of confidence also increases the VaR. (Danielsson, 2011).

Lock-up Period: It shows how long the assets will stand on the market and this ratio is directly proportional to VaR. The longer the duration, the higher the expected price change will be (Keasler, 2001, pp.214).

Distribution of Financial Assets: The assumption is made that the returns of the assets for the VaR account distributed normally. However, it may not be normally distributed in practice.

Portfolio Diversification: Investors can reduce their risks by diversifying their portfolios. The method that can be used at this stage is the "Average Variance Model" proposed by Markowitz (Rubinstein, 2002, pp.1042). The value at risk is simply calculated by the following formula;

$$VaR = P \cdot z_{\alpha} \cdot \sqrt{T} \cdot s$$

P : The value of the portfolio, T : lock-up period, z_{α} : confidence level, s : standard deviation of returns. The value-at-risk calculation methods commonly used in the literature can be summarized as follows.

3.1. Delta Normal Method

It is the simplest calculation method known. Under the assumption of normality, VaR is calculated using z_{α} values.

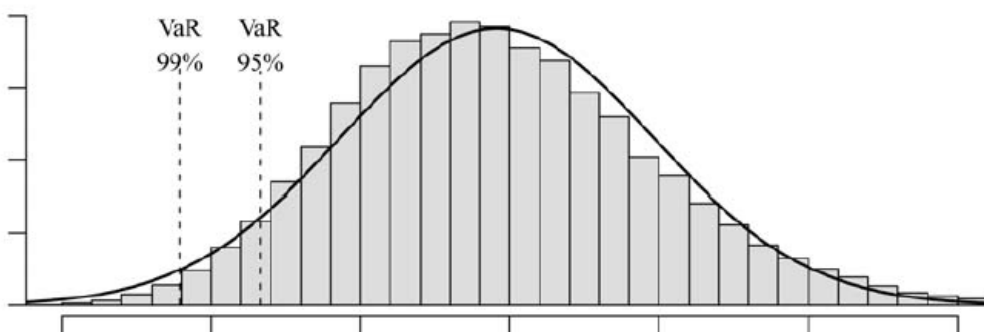


Figure 1. Under the assumption of normality VaR values (Danielsson, J., 2011)

3.2. Historical Simulation Method

In this method, scenarios are produced from historical market data. The portfolio is valued by using historical changes in risk factors. Accordingly, the profit/loss distribution of the portfolio is calculated. In this model, there is no assumption that

returns are normally distributed. There is no need to calculate volatility, correlation or other parameters. The probability of model risk is very low. The lack of historical simulation method is the complete neglect of the cases not reflected in the data set used. (Van den Goorbergh and Vlaar, 1999, pp.20).

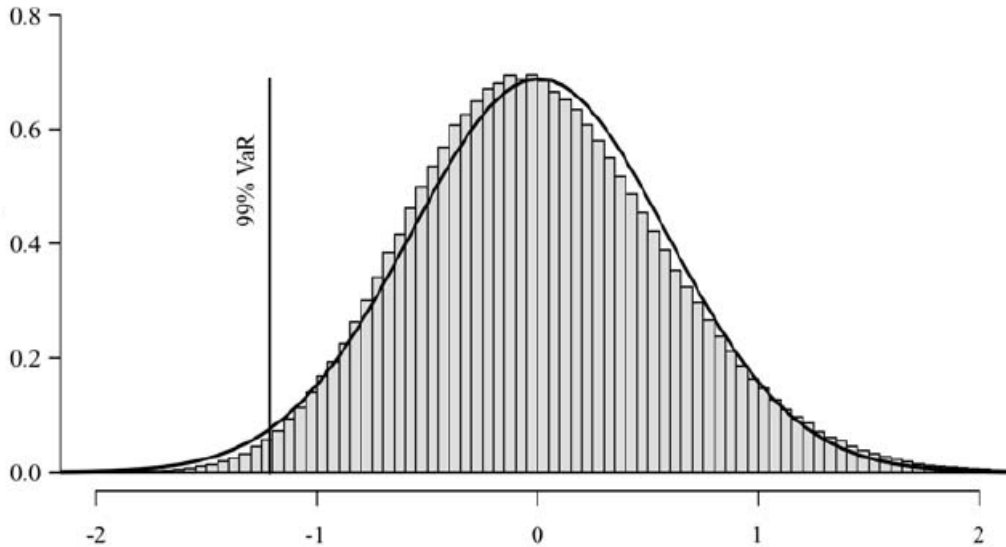


Figure 2. VaR Value for Historical Simulation (Danielsson, J., 2011)

3.3. Monte Carlo Simulation Method

This method is similar to the historical simulation method. But the difference of Monte Carlo is that the scenarios are derived from a determined distribution rather than actual historical data. In Monte Carlo simulation, simulated data is used by selecting a statistical distribution that reflects the possible changes in prices (Danielsson, 2011). It is known as the most comprehensive and strongest value-at-risk calculation method. However, in the estimation, the variance is large and is the subject of discussion as an effective estimation method.

3.4. Importance Sampling

Although the MC method is important, it is not effective. By using importance sampling method, the variance is reduced, and a more efficient estimator is obtained.

Glasserman (2003) mentioned the theoretical structure of importance sampling. Here, it is worth mentioning the basic elements of this variance reduction method. Let X be a random variable with probability density of f and defined in \mathbb{R}^d . Assume that there is a function defined in $h: \mathbb{R}^d \rightarrow \mathbb{R}$. In this case, by the definition of expected value, the value of $E[h(X)]$ can be found as follows;

$$I = E[h(X)] = \int h(x)f(x)dx \tag{1}$$

In this case, the standard Monte Carlo estimator is,

$$\hat{I}_{MC} = \frac{1}{N} \sum_{i=1}^N h(X_i) \tag{2}$$

(Glasserman, 2003). Here, X_i observations are independent samples drawn from the density of f . This estimator is not effective in most cases. Therefore, the sampling

efficiency should be increased by giving more weight to the “significant” results by using a change in size (Glasserman, 2003).

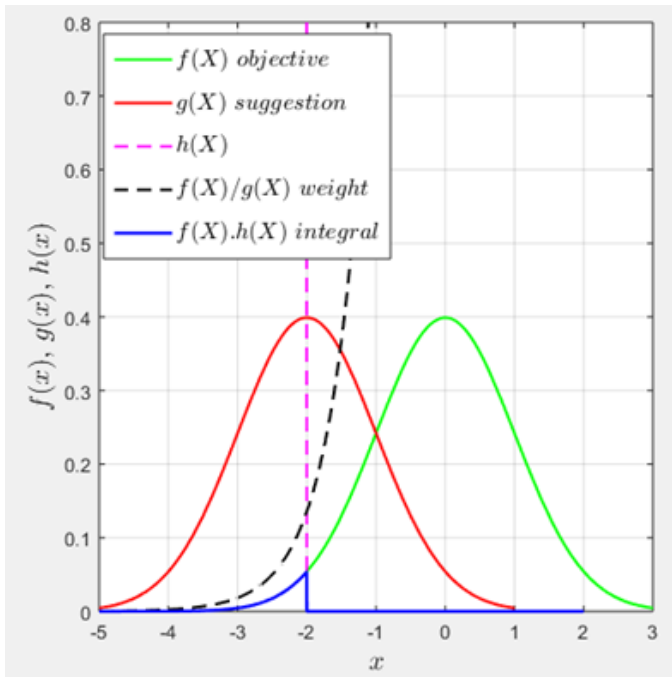


Figure 3. Importance Sampling

Defined in \mathbb{R}^d for $\forall x \in \mathbb{R}^d$, considering a new probability density function of g (importance sampling distribution) that provides the condition $f(x) > 0 \Rightarrow g(x) > 0$, Equation (1) can be written as follows;

$$I = \int h(x) \frac{f(x)}{g(x)} g(x) dx \tag{3}$$

This final equation obtained can be interpreted as the expected value under the distribution with the density of g . Thus, if equation (3) is written by using the expected value operator, equation (4) is found:

$$I = E \left[h(X) \frac{f(X)}{g(X)} \right] \tag{4}$$

Random variable X in equation (4) has a density of g . If X_1, X_2, \dots, X_N are independent random variables drawn from g a distribution, importance sampling estimator associated with g is defined as ;

$$\hat{I}_{IS} = \frac{1}{N} \sum_{i=1}^N h(X_i) \frac{f(X_i)}{g(X_i)} \tag{5}$$

(Glasserman, 2003). Here, the weight of $f(X_i)/g(X_i)$ is called *the* likelihood ratio or derivative of Radon-Nikodym at the point X_i (Glasserman, 2003). It is obvious to see that $E[\hat{I}_{IS}] = I$. The asymptotic variance of the estimator \hat{I}_{IS} is defined in equation (6).

$$Var(\hat{I}_{IS}) = \frac{\sigma_h^2}{N} = \frac{1}{N} \left\{ \int \frac{h^2(x) f^2(x)}{g(x)} dx - I^2 \right\} \quad (6)$$

The distribution of g suggestion to be selected directly affects the success of importance sampling. Because to reduce the variance in equation (6), it has to be $g(x) \propto h(x)f(x)$ (Glasserman, 2003). Otherwise, $g(x)$ shows a slower convergence than the numerator of equality (6), $Var(\hat{I}_{IS}) \rightarrow \infty$. This situation is very far from the purpose. One way to ensure that this does not happen is to select a $g(x)$ sampling distribution with the non-zero domain, where $f(x)$ has the non-zero definition set. In this case, choosing a $g(x)$, ($f(x) \leq kg(x)$, $k \in \mathbb{R}$) that covers the $f(x)$ guarantees that the variance does not increase.

In this study, descriptive statistics of the S&P index between 1993 and 2003 were presented and the methods mentioned above were applied. The simulation trial numbers for the data are determined as {10, 100, 1000, 10000}. Confidence levels and lock-up periods were determined respectively as {99%, 95%} and {1,10} days. The necessary codes and calculations for the study were written in MATLAB r2018a and the results were reported.

4. Analysis and Results

The histogram of the S&P500 index between 1993-2003 is shown in Figure 4. Some descriptive statistics of these data are presented in Table 1.

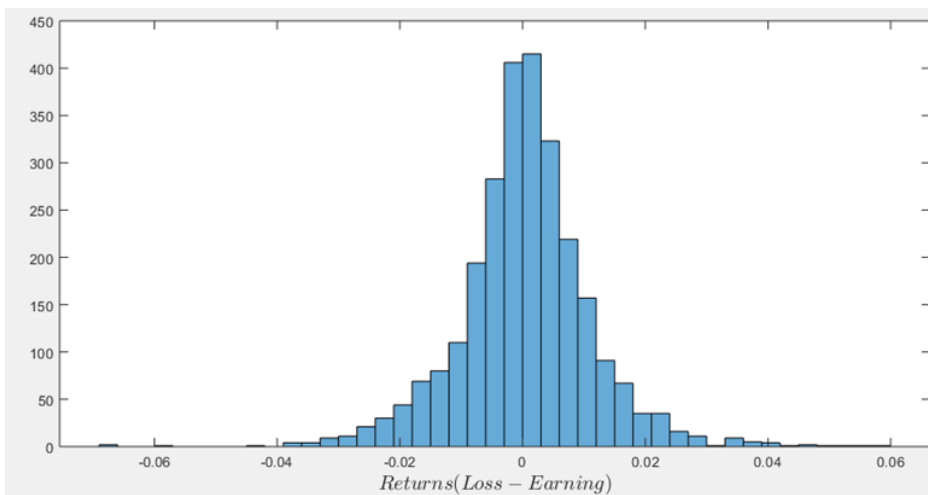


Figure 4. Histogram of the S&P 500 Index between 1993-2003

Min.	-0.06866	Average	0.00037
Maks.	0.05732	Mode	0.00038
1.Quartile	-0.00492	Std.Deviation	0.01087
2.Quartile	0.00034	Skewness	-0.01938
3.Quartile	0.00586	Kurtosis	6.43933

Table 1. Some descriptive statistics for the S&P 500 index

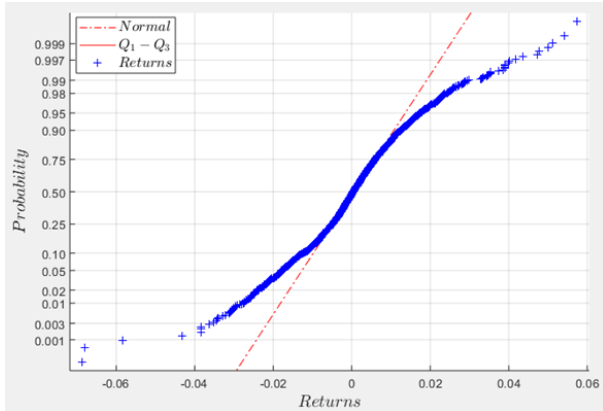


Figure 5. Graph of Normality for Returns

When the descriptive statistics and normality graphs are examined, it is clearly seen that the data is not normally distributed and that it occurs in a leptokurtic structure. In this case, the methods mentioned in the third section are applied to these data. The results of the delta normal method for a portfolio value of 5000 units are shown in Table 2 and the results for the historical simulation method are shown in Table 3. When Table 2 is examined, the maximum amount that the investor will lose in 1 day at a 99% confidence level (VaR) is 126.76 units, the maximum amount to lose for 1 day at a 95% confidence level is 89.46 units.

	%99	%95
$T=1$	126.70637	89.45578
$T=10$	400.68074	282.88406

Table 2. VaR Values for Delta Normal Method

	%99	%95
$T=1$	138.04435	89.50708
$T=10$	436.53456	283.04625

Table 3. VaR Values for Historical Simulation Method

In the historical simulation method, the calculation is based on the pth-quartile since there is no distribution assumption. The data is sorted from the minimum return to the highest return and the VaR values are calculated by the corresponding trust level. When the VaR values calculated by this method are examined, the maximum amount to loose in 1 day at a 99% confidence level is 138.04 units, while the maximum amount to loose for 1 day at a 95% confidence level is 89.51 units. The sample size (n) was set to 100 in each simulation and the portfolio value was determined as 5000 units for Monte Carlo simulation. To perform the simulation, it is necessary to make the distribution assumption in this method. Therefore, throughout all simulation, data is derived from the normal distribution with an average of 0 and variance 1, and calculations are made by the equation (2).

N	10	100	1000	10000
\hat{I}_{MC}	0.02600	0.02210	0.02200	0.02226
$Var(\hat{I}_{MC})$	0.00013	0.00019	0.00020	0.00025
$VaR_{t=1}^{MC}$	105.66842	109.42369	109.32067	108.82248
$VaR_{t=10}^{MC}$	334.15291	346.02809	345.70232	344.12690

Table 4. VaR Values for Monte Carlo Method

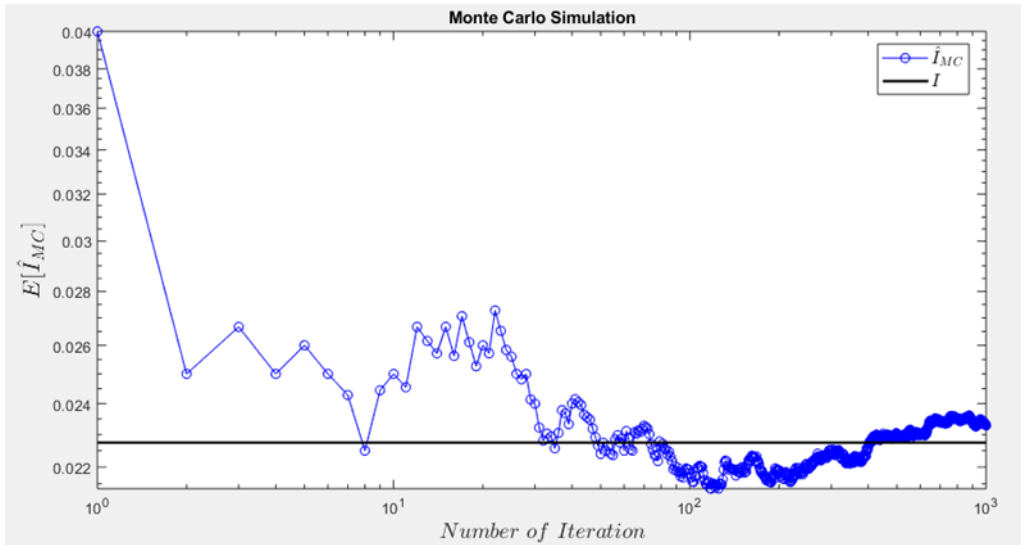


Figure 6. Monte Carlo Simulation

When Table 4 is examined, in 1-day VaR of the investor at the 99% confidence level is 109.32 and 108.82 units for $N = 1000$ and 10000 values, respectively. Also, when the number of simulations increases, the variance of the Monte Carlo estimator increases. A similar algorithm is followed for importance sampling. However, an assumption of "suggestion($g(x)$)" distribution is required, which is considered to represent rare events. To emphasize more rare events, the $g(x)$ distribution is chosen to be normally distributed, with a mean of 2 and a variance of 1. $h(x)$ is the indicator function. Figure-7 shows objective function $f(x)$, suggestion $g(x)$, indicator function $h(x)$ and convergence to $E[\hat{I}_{IS}]$. The VaR values for the 99% confidence level using these arguments are summarized in Table 5. By using importance sampling, the investor's maximum losses for 1 day at 99% confidence levels are calculated as 108.73 and 108.75 units respectively for $N = 1000$ and 10000 values. Since the variance values obtained during the simulations are about 10 times smaller than in Table 5. Besides, the MC estimator can achieve a good convergence at 10000 iterations, while the IS estimator achieves this convergence about 10 times fewer iterations. Therefore, it is also efficient in terms of transaction volume.

N	10	100	1000	10000
\hat{I}_{IS}	0.02106	0.02224	0.02277	0.02275
$Var(\hat{I}_{IS})$	0.000014	0.000012	0.0000113	0.0000101
$VaR_{t=1}^{IS}$	110.51593	109.01598	108.73225	108.75535
$VaR_{t=10}^{IS}$	349.48206	344.73880	343.84157	343.91461

Table 5. VaR Values for Importance Sampling

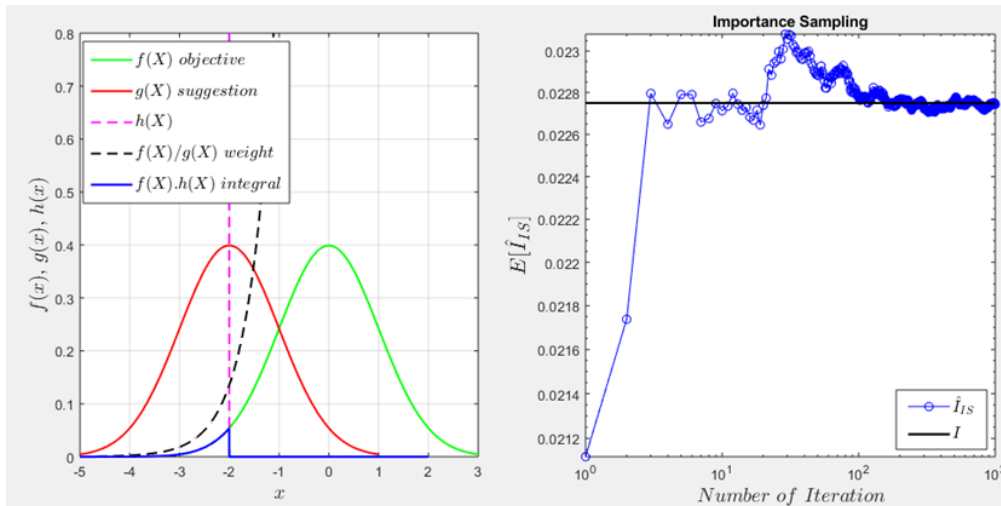


Figure 7. Importance Sampling

The key to risk measurement is to consider the measurements that can yield truly effective results, not VaR results.

5. Conclusion and Suggestions

Financial institutions, banks, and investment advisors are expected to make the best of the risk management by considering the conditions of the market they are in. Nowadays, there are rare events in the finance field that disrupt the general structure. Rare events can be not only the left tails but also the right tails of the distributions. In such cases, it is important to reflect the real risk to the investor. One of the methods of risk measurement is the Value at Risk (VaR) method. In this study, the usage of delta normal method, historical simulation method, Monte Carlo simulation and importance sampling to calculate the Value at Risk and to show which method is more effective by applying them to the S&P index between 1993 and 2003. The Delta normal method is very easy to calculate, but in most cases the data is not normally distributed, so it gives very misleading results in risk measurement. The historical simulation method can be a realistic method for the investor since it is easy to apply and real historical data is used instead of generating random numbers due to the scenario. Because it gives weight to each data at the same rate and the difference of big changes cannot be seen. What we would like to address in this study is the effect of importance sampling on risk estimation. In this context, Monte Carlo method and importance sampling method were compared.

N	n	$Var(\hat{I}_{IS})$	$Var(\hat{I}_{MC})$
10	100	0.000014	0.00013
100	100	0.000012	0.00019
1000	100	0.0000113	0.00020
	200	0.0000068	0.0001014
10000	100	0.0000101	0.00025
	200	0.0000060	0.0001123

Table 6. Variance of Estimators According to n, N

In the empirical application, it is found that \hat{I}_{IS} converges to the parameter value almost 10 times faster than the \hat{I}_{MC} . It is clear in Table 6 that as simulations increase for sample size $n=100$ and $n=200$, variance of Monte Carlo estimator increase and variance of importance sampling estimator decrease. Another significant result is obtained with comparing variance values of estimators during simulations with different sample sizes. According to the variance values obtained from the simulations, the variance of the importance sampling estimator is approximately 10 times smaller than the variances of the Monte Carlo estimator. Considering the concept of risk and its importance in the financial literature, the difference, which is approximately 10 times between variance values, is quite significant. Thus, it is shown that the importance sampling method reduces the variance of loss probabilities and produces asymptotically effective results from the Monte Carlo simulation.

Therefore; in this study, it is proposed the importance sampling method to reduce variance of loss probabilities and so the risk. In this way, the investor will be freed from the misleading statistical value and thus the loss, in addition, to make a rational decision.

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